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Anisotropy in the flow processes of a thin plastic layer☆

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Abstract

The theory of the flow of a thin layer of plastic material over surfaces developed by Il'yushin is extended to the case of an anisotropic ideally plastic material and anisotropic flow on the surface. Particular attention is given to determining the contact pressure. Two methods are proposed for solving this problem: a variational method and reduction to a Cauchy problem. The effect of anisotropy is revealed using specific examples.

A theory of the flow of thin layers of plastic material over surfaces was proposed by Il'yushin.[1,2] This theory was subsequently developed further and has found numerous applications. All of these investigations refer to the case of an isotropic material and isotropic friction on the contact surfaces. Meanwhile, in current pressure treatment technology, there is considerable interest in processes which take account of the anisotropy of both the material as well as the contact friction.[3–6] In this paper, Il'yushin's theory is extended to both cases of anisotropy. It is shown that in both cases the pressure in a layer is determined by mathematically identical equations, but, on the other hand, the velocity fields are basically different. In the case of an anisotropic material, we assume that the degree of deformation of the layer is small, so that the type and the magnitude of the anisotropy can be assumed to be unchanged. This constraint is not imposed in the case of anisotropic friction.

1. The compression of a strip by rigid parallel plates

The analysis of the cycloidal Prandtl solution[7] served as the basis for formulating the hypotheses of the theory in Refs. 1,2. We will now extend this solution to the case of a material, the properties of which are described (in the principal axes of anisotropy) by the quadratic Mises-Hill form[8]

$$F(\sigma_y-\sigma_z)^2+G(\sigma_z-\sigma_x)^2+H(\sigma_x-\sigma_y)^2+2L\tau_{yz}^2+2M\tau_{zx}^2+2N\tau_{xy}^2 = 1 \quad (1.1)$$

The coefficients of the form are connected by known relations with the yield points for stretching and shear along the principal directions.

Consider a layer which occupies the domain

$$S: \{|x|\le l,\ |y|\le h,\ |z|<\infty\}$$

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The rigid planes $x=\pm h$ move with a velocity $\mp v_0$ respectively. Under conditions of plane deformation

$$\sigma_z = (G\sigma_x + F\sigma_y)/(G+F), \quad \tau_{yz} = \tau_{zx} = 0$$

and the stresses σ_x, σ_y, τ_{xy} are found from the two equilibrium equations and the plasticity condition (1.1), which takes the form

$$\frac{(\sigma_x - \sigma_y)^2}{4(1-c)} + \tau_{xy}^2 = \tau_s^2, \quad \frac{1}{\tau_s^2} = \frac{1}{T^2} = 2N$$
$$c = 1 - \frac{N(F+G)}{2(FG+GH+HF)}, \quad -\infty < c < 1 \tag{1.2}$$

where τ_s is the shear yield point in the direction of the x-axis.

If it is assumed that, subject to the condition $(h/l) \ll 1$, the difference $\sigma_x - \sigma_y$ and the stress τ_{xy} are independent of x, then, following the well-known procedure in Ref. 8, we obtain

$$\sigma_x = -\tau_s\frac{l-x}{h} + 2\tau_s\sqrt{1-c}\left(\sqrt{1-\frac{y^2}{h^2}} - \frac{\pi}{4}\right)$$
$$\sigma_y = -\tau_s\frac{l-x}{h} - 2\tau_s\sqrt{1-c}\frac{\pi}{4}, \quad \tau_{xy} = -\tau_s\frac{y}{h} \tag{1.3}$$

As a consequence of symmetry, this solution is written for the first quadrant, and the boundary condition on the edge $x=l$ is satisfied integrally.

The velocity field is found from the incompressibility condition and the law of plastic flow

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad 2(1-c)\tau_{xy}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) = (\sigma_x - \sigma_y)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\right)$$

Assuming $v = v(y)$ and taking account of relations (1.3), we hence obtain

$$v = -v_0\frac{y}{h}, \quad u = \frac{v_0 x}{h} + 2v_0\sqrt{1-c}\left(\sqrt{1-\frac{y^2}{h^2}} - \frac{\pi}{4}\right) \tag{1.4}$$

The solution (1.3), (1.4) possesses all the properties of the Prandtl solutions, and its analysis leads to the same well-known conclusions.[1] All this provides a basis for adopting the hypotheses in Ref. 1,2 when formulating the problem of the flow of a thin plane layer over rigid surfaces.

2. Formulation of the problem for the case of an anisotropic material

Consider a thin layer of plastic material with the properties (1.1). It occupies a domain S_0 in the x, y plane bounded by a piecewise-smooth contour Γ. The layer is bounded by the surfaces $z=f_1(x, y, t)$ and $z=f_2(x, y, t)$ such that the difference $h(x, y, t)=f_2-f_1$ defines the thin layer as a known function of the coordinates and time. It is important that h is small and a smoothly changing function of the coordinates and time. We will consider quasistatic processes and we shall therefore henceforth omit the time as an argument.

The vector of the velocity of relative slippage of the material and the surfaces f_2 and f_1 is denoted by v (we assume that there are no so-called internal motions[1] of the surfaces); $v = |v| \cdot \mathbf{n}^\circ$, $\mathbf{n}^\circ = \{\cos\theta, \sin\theta\}$. We will assume that the friction conditions on the two surfaces are the same and we will denote the magnitude of the friction force by τ_m. In extending Prandtl's hypothesis and the results in Refs. 1,2, we will assume that the friction force τ_m is equal to the yield point of the material in the direction $\mathbf{n}^\circ$. If the coordinate axes coincide with the principal directions of anisotropy, it follows from relations (1.1) that

$$\tau_{s\theta}^2 = \left(\frac{\sin^2\theta}{R^2} + \frac{\cos^2\theta}{S^2}\right)^{-1} \tag{2.1}$$

Here, $S=(2M)^{-1/2}$ and $R=(2L)^{-1/2}$ are the shear yield points in the x and y directions respectively. To be specific, we shall assume that $S>R$, in which case

$$S^2/R^2 = \beta^2 = 1+\delta, \quad \delta>0$$

Putting $S=\tau_s$, from relation (2.1) we obtain

$$\tau_{s\theta} = \frac{\tau_s}{(1+\delta\sin^2\theta)^{1/2}} \tag{2.2}$$

We will denote the pressure from the side of the layer on the surface by $p=-\sigma_z$; the stresses σ_x and σ_y differ by an amount of the order of max h/l, where l is the characteristic dimension of S_0. From condition (1.1), we therefore obtain

$$P+\sigma_x \cong p+\sigma_y = (F+G)^{-1} = \frac{1}{Z} \equiv \sigma_s \tag{2.3}$$

The contact friction stresses are directed opposite to the direction of $\mathbf{n}°$, and the equilibrium equations

$$\operatorname{grad} p = -\frac{2\tau_s}{h(x,y)(1+\delta\sin^2\theta)^{1/2}}\mathbf{n}° \tag{2.4}$$

follow from this.

Apart from terms of the order of max (h/l), the boundary conditions on the contour Γ have the form $\sigma_x=\sigma_y=0$, and, from relations (2.3), we therefore obtain

$$x, y \in \Gamma, \quad p = \sigma_s \tag{2.5}$$

We now refer the coordinates to l, while retaining the previous notation for them, and assume that

$$h = h_0 g(x,y), \quad (p-\sigma_s)h_0/(2\tau_s l) = z$$

It follows from relations (2.4) and (2.5) that

$$\operatorname{grad} z = -\frac{\mathbf{n}°}{g(x,y)(1+\delta\sin^2\theta)^{1/2}} \tag{2.6}$$

$$x, y \in \Gamma, \quad z = 0 \tag{2.7}$$

Since $\cos\theta = u/|v|$ and $\sin\theta = v/|v|$, it is necessary to supplement system (2.6), (2.7) with an incompressibility equation which, in dimensionless variables, takes the form

$$\frac{\partial(gu)}{\partial x} + \frac{\partial(gv)}{\partial y} = \frac{lv_0}{h_0 g} \tag{2.8}$$

Here v_0 is the velocity of approach of the surfaces.

System (2.6), (2.7), (2.8) is separated. In fact, we obtain from (2.6) that

$$\left(\frac{\partial z}{\partial x}\right)^2 + \beta^2\left(\frac{\partial z}{\partial y}\right)^2 = \frac{1}{g^2(x,y)} \tag{2.9}$$

If, subject to condition (2.7), a function $z(x, y)$ is found from this, then, for u and v, we have a system of Eq. (2.8) and the relation

$$u\frac{\partial z}{\partial y} = v\frac{\partial z}{\partial x} \tag{2.10}$$

We shall pay particular attention to problem (2.9), (2.7).

3. The general solution and some examples

We now make the replacement of variables $y=\beta\eta$ in Eq. (2.9) and obtain

$$\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial \eta}\right)^2 = \frac{1}{g^2(x, \beta\eta)} \equiv \frac{1}{\psi^2(x, \eta)} \tag{3.1}$$

If $y=\varphi(x)$ is an explicit equation of the contour Γ, we shall have $\Gamma \Rightarrow \Gamma'$: $\eta=\varphi(x)/\beta$ in the new variables. Suppose $z(x, \eta)=c=\text{const}$ is the family of level lines. Then, at least in the neighbourhood of the contour and in the domain where it is smooth, a family γ of orthogonal trajectories can be constructed. We will denote one of the lines of this family which passes through the point x_0, η_0 of the contour by $\eta=f_0(x, x_0, \eta_0)$. As the parameter for γ, we will choose the length of an arc and measure it from the contour. It follows from equation (3.1) that $dz/ds=1/\psi(x, \eta)$, whence

$$z = \int_\gamma \frac{ds}{\psi(x, \eta)} = \int_{x_0}^{x_1} \frac{(1+\eta'^2)^{1/2}dx}{\psi(x, \eta)} \equiv \int_{x_0}^{x_1} F(x, \eta, \eta')dx \tag{3.2}$$

Hence, if the family γ is known, the pressure is found by simple quadrature.

We will now show that the lines γ coincide with the extremals of the variational problem for the functional (3.2). The proof is based on the replacement of the Euler-Lagrange equation

$$\frac{d}{dx}F'_{\eta'} - F'_{\eta} = 0 \tag{3.3}$$

by a system of canonical equations using the Legendre transformation $F'_{\eta'}=q$:

$$\frac{dq}{dx} = -\frac{\partial L}{\partial \eta}, \quad \frac{d\eta}{dx} = \frac{\partial L}{\partial q}; \quad L(x, \eta, q) = q\eta' - F(x, \eta, q) \tag{3.4}$$

Eqs. (3.4) are the characteristic system for the equation

$$\frac{\partial z}{\partial x} + L\left(x, \eta, \frac{\partial z}{\partial \eta}\right) = 0 \tag{3.5}$$

In the case being considered

$$q = F'_{\eta'} = \eta'(1+\eta'^2)^{-1/2}/\psi'$$

whence

$$\eta'^2 = \psi^2 q^2(1-\psi^2 q^2)^{-1}, \quad L = -(1-\psi^2 q^2)^{1/2}/\psi$$

Substituting the resulting expression for L, replacing q by $\partial z/\partial\eta$, into relation (3.5), we obtain Eq. (3.1). The transversality condition for the functional (3.2) implies orthogonality. The proof is thereby completed.

Eq. (3.3) is a second order differential equation

$$\psi\eta'' + (1+\eta'^2)\left(\frac{\partial \psi}{\partial \eta} - \frac{\partial \psi}{\partial x}\eta'\right) = 0 \tag{3.6}$$

the solution of which depends on two parameters: $\eta=f(x, a, b)$. The following four conditions enable us to find these parameters and the point x_0, η_0, if a point x, η, through which the trajectory of the family γ passes, is specified,

$$f(x, a, b) = \eta, \quad \varphi(x_0) = \beta\eta_0, \quad \varphi(x_0) = \beta f(x_0, a, b), \quad \varphi'(x_0)f'(x_0, a, b) = -\beta$$

Example 1. When $\psi=1$, we have a problem on the compression of a layer by parallel plates. From Eq. (3.6), we find the family γ as a two-parameter family of lines which are orthogonal to the contour (similar to a sandy embankment,

A. A. Il'yushin, 1954). The analytical solution of the problem can be easily written out. If $x_0 = x_0(t)$ and $\eta_0 = \eta_0(t)$ are the parametric equations of the contour, the equation of the normal to it has the form

$$\eta_0 - \eta = -(x_0'/\eta_0')(x_0 - x) \quad (3.7)$$

The pressure is determined as the distance along the normal (3.7) from the contour to the current point x, η:

$$z = (x_0 - x)[1 + x_0'^2/\eta_0']^{1/2} \quad (3.8)$$

In order to obtain z as an explicit function of x and η, from the equality (3.7) it is necessary to express t as a function of x and η and substitute this into expression (3.8).

The first elementary solution is obtained for the circular domain

$$x_0^2 + \eta_0^2 = 1, \quad x_0 = \cos t, \quad \eta_0 = \sin t$$

From equality (3.7) we find $\operatorname{tg} t = \eta/x$ and from expression (3.8) we find $z = 1 - \sqrt{x^2 + \eta^2}$. In the initial formulation, this is the flow in the elliptic domain $x_0^2 + y_0^2/\beta^2 = 1$; for the pressure we obtain $z = 1 - \sqrt{x^2 + y^2/\beta^2}$, and the pressure diagram is an elliptic cone $(1 - z)^2 = x^2 + y^2/\beta^2$.

The level lines are the ellipses

$$x^2/(1-c)^2 + y^2/(\beta^2(1-c)^2) = 1$$

As can be seen from Eq. (2.6) the streamlines are orthogonal trajectories with respect to the level lines. In the example considered, the calculations lead to the result that the parabolae $y = y_0(x/x_0)^{1/\beta^2}$ are streamlines passing through the point x_0, y_0 of the contour.

We present a second simple solution for a rectangular domain $|x| \le a$, $|y| \le b$ solely to show that this is a unique case when the level lines and the streamlines are a grid of straight lines parallel to the coordinate axes, and the solution of the initial problem is given within the framework of the sand analogy.

Example 2. Eqs. (2.9) and (2.7) constitute the Cauchy problem: it is required to find an integral surface $z(x, y)$ which passes through the contour Γ in the x, y plane. We put $g = 1$ and construct a solution based on the fact that the total integral of (2.9) is known

$$z + \sqrt{1 - \beta^2 a^2}\, x + ay - b = 0 \quad (3.9)$$

We will specify the Cauchy conditions in parametric form: $z = 0$, $x = \varphi(t)$, $y = \psi(t)$ and, on substituting these into equality (3.9), we obtain

$$\sqrt{1 - \beta^2 a^2}\, \varphi(t) + a\psi(t) - b = 0$$

The additional equation for determining the parameters a and b is found from the relation

$$\sqrt{1 - \beta^2 a^2}\, \varphi'(t) + a\psi'(t) = 0$$

Finally, equality (3.9) takes the form

$$z = (-\psi' x + \varphi' y + \varphi\psi' - \varphi'\psi)(\psi'^2 + \beta^2\varphi'^2)^{-1/2} \quad (3.10)$$

The parameter t as a function of x and y is eliminated from the equation $z_t' = 0$.

We now consider flow in the circle $\varphi = \cos t$, $\psi = \sin t$. It follows from relation (3.10) that

$$z = (1 - x\cos t - y\sin t)(1 + \delta\sin^2 t)^{-1/2} \quad (3.11)$$

The condition $z_t' = 0$ leads to the equation

$$\beta^2 x \sin t - y\cos t = \delta \sin t \cos t \quad (3.12)$$

which reduces to a complete fourth order algebraic equation in $\sin t$, which is not suitable for an analytical investigation. Eq. (3.12) has a unique solution in the sector $x^2+y^2 \le 1$, $0 \le t \le \pi/2$, and there is no difficulty in finding this solution using numerical methods.

If the degree of anisotropy is small such that $\delta^2 \ll 1$, the solution of Eq. (3.12) can be found by expansion with respect to a small parameter and, in the first approximation, we obtain

$$\operatorname{tg} t \cong (y/x)(1+\delta\alpha), \quad \alpha = 1/r - 1, \quad r^2 = x^2 + y^2$$

After substitution in to expression (3.11), we will have

$$z \cong \left[1 - \frac{x^2 + (1+\delta\alpha)y^2}{(r^2 + 2\delta\alpha y^2)^{1/2}}\right]\left(1 + \frac{\delta y^2}{r^2}\right)^{-1/2}$$

Some other examples of the solution of Eq. (3.1) can be found in Ref. 9. Analogies between this problem and various problems in geometry and physics are also pointed out in Ref. 9.

4. Anisotropic friction

The assumption that the contact friction stress is in the opposite direction to the relative velocity of slippage vector, traditionally used in mathematical models of processes involving treatment with pressure. At the same time, for the purpose of controlling a process, the contacting surfaces can be finished with a particular texture or thin lubricating layers with anisotropic properties can be applied. In both cases, we can take[6]

$$\tau^\circ = \frac{\tau_m}{|\tau_m|} = -A\mathbf{n}^\circ$$

for the shear stress vector τ_m.

In the case being considered, the surfaces f_1 and f_2 are characterized by the matrices

$$A_1 = \{\alpha'_{ik}\}, \quad A_2 = \{\beta'_{ik}\}; \quad i, k = 1, 2$$

Without loss of generality, the matrices A_1 and A_2 can be assumed to be diagonal matrices: $\alpha'_{ik} = \alpha_{ik}\delta_{ik}$, $\beta'_{ik} = \beta_{ik}\delta_{ik}$ and the principal directions of anisotropy can be assumed to be identical on the two surfaces. Then, in agreement with well-known results,[2] the equilibrium equations of the layer take the form (subject to the condition $|\tau_m| = \tau_s$)

$$\frac{\partial p}{\partial x} = -\frac{(\alpha_{11}+\beta_{11})\tau_s}{h(x,y)}\cos\theta, \quad \frac{\partial p}{\partial y} = -\frac{(\alpha_{22}+\beta_{22})\tau_s}{h(x,y)}\sin\theta \tag{4.1}$$

We put $\alpha_{11}+\beta_{11} = \alpha$, $\alpha_{22}+\beta_{22} = \mu\alpha$.

As above, we introduce dimensionless coordinates and the layer thickness and use the notation

$$(p-\sigma_s)h_0/(\alpha\tau_s l) = z$$

From Eq. (4.1), we obtain

$$\frac{\partial z}{\partial x} = -\frac{\cos\theta}{g(x,y)}, \quad \frac{\partial z}{\partial y} = -\frac{\mu\sin\theta}{g(x,y)} \tag{4.2}$$

To fix our ideas, we shall assume that $\mu < 1$ and put $\mu^2 = 1/\beta^2$. From Eq. (4.2), we then obtain

$$\left(\frac{\partial z}{\partial x}\right)^2 + \beta^2\left(\frac{\partial z}{\partial y}\right)^2 = \frac{1}{g^2(x,y)}$$

This equation is identical to (2.9). Together with the condition $z|_\Gamma = 0$, we obtain the problem of determining z, which is identical to that considered in the preceding section. However, we note that the velocity field in this case is quite different from that considered above: the streamlines and the level lines are not orthogonal and, to determine υ, we have the equation of continuity (2.8) and the relation

$$u\frac{\partial z}{\partial y} = \mu\upsilon\frac{\partial z}{\partial x} \tag{4.3}$$

which differs from (2.10).

5. The general case of anisotropy

If the principal directions of anisotropy in the properties of a material and of the matrices A_1 and A_2 coincide, the preceding results can be generalized in an obvious way: to determine the pressure, we obtain

$$\frac{\partial z}{\partial x} = -\frac{\cos\theta}{g(1+\delta\sin^2\theta)^{1/2}}, \quad \frac{\partial z}{\partial y} = -\frac{\mu\sin\theta}{g(1+\delta\sin^2\theta)^{1/2}}$$

whence

$$\left(\frac{\partial z}{\partial x}\right)^2 + \frac{\beta^2}{\mu^2}\left(\frac{\partial z}{\partial y}\right)^2 = \frac{1}{g^2(x,y)}$$

The velocity field is determined from relations (2.8) and (4.3).

Note the degenerate version $\beta = \mu$ when the problem of determining the magnitude of z is transformed into the well-investigated case: $|\text{grad}\, z|^2 = g^{-2}$, $z|_\Gamma = 0$.

Remark. Everything what has been described above refers to the case of a fixed domain. Meanwhile, the problem of spreading, that is, the problem of determining the shape of the domain which the layer occupies after a specified compression (the initial shape is known), is of considerable interest, particularly in the case of anisotropic friction. There are serious mathematical difficulties associated with the formulation of this problem, and its solution is awaited.

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